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Secondary polytope, hypergeometric D-module and connection formulas of $\Delta_1 \times \Delta_{n-1}$-hypergeometric functions *

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Summary: Recently, the study of hypergeometric functions has popped up in diverse quarters as the study of moduli spaces of algebraic varieties, binomial sums, conformal field theory, statistical mechanics and so on. In this exposition, we focus on the following topics;

1. A history of the study of hypergeometric functions.
2. Solution sheaf of A-hypergeometric system and the secondary polytope.
3. Derivation of the connection formulas of $\Delta_1 \times \Delta_{n-1}$-hypergeometric function as an application of the result 2.

1. A history — from Gauss-schwartz theory

Why are the hypergeometric functions interesting? I start with trying to explain a reason. I believe that it is the best to show you the Gauss-schwartz theory in order to answer to the question.

Let

$$y^2 = x(x-1)(x-t), \quad t \neq 0, 1, \infty$$

be a family of the elliptic curves $\{C_t\}$. Since the genus of C_t is 1, the space of holomorphic 1-forms $H^0(C_t, \Omega^1)$ is 1-dimensional vector space; the 1-form

$$\frac{dx}{y} = x^{-1/2}(x-1)^{-1/2}(x-t)^{-1/2}dx$$

spans the space of holomorphic 1-forms on C_t. Let α_t and β_t be the generators of the homology group $H_1(C_t, \mathbf{Z})$ with the intersection matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. For example, you may take α_t and β_t as in the figure 1. The period map of $\{C_t\}$ is

$$p \ : \ \mathbf{C} \setminus \{0, 1, \infty\} \ni t \longmapsto \left(\int_{\alpha_t} \frac{dx}{y}, \int_{\beta_t} \frac{dx}{y} \right) = (p_1(t), p_2(t)).$$

One of the main problem in the algebraic geometry is the classification of algebraic varieties. The period map plays an important role in this problem by virtue of Torelli's theorem.

* In "modern mathematics and polyhedral geometry" edited by T.Hibi (RIMS Kokyuroku, Kyoto Univ.)

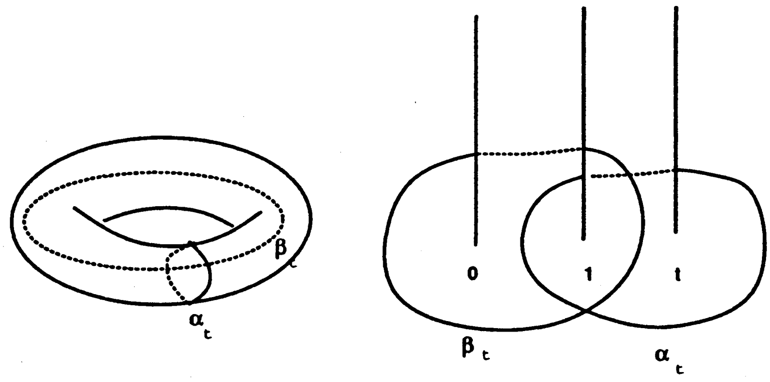

Figure 1.

THEOREM (Torelli, see [G; 173p]) *A necessary and sufficient condition for two compact Riemann surfaces C and C' be isomorphic, is that they have the same normalized period matrix under a suitable choice of canonical homology bases.*

In our example, the normalized period matrix is $(1, p_2(t)/p_1(t))$. In order to get the moduli space through the period map, we must determine the image of the upper and lower half plane by the map $p_1(t)/p_2(t)$. How to get the image? We can get the image by studying the local and global behaviors of the period map. The key role is played by the Gauss hypergeometric function.

PROPOSITION 1.1. *Assume the configuration of Figure 1 and* $\mathrm{Im}\, t > 0$. *We have*

$$p_1(t) = -2i\Gamma(\frac{1}{2})^2 F(1/2, 1/2, 1; 1-t)$$
$$p_2(t) = -2\Gamma(\frac{1}{2})^2 t^{-1/2} F(1/2, 1/2, 1; 1/t)$$

where we regard z^α as the single valued function on $\mathbf{C} \setminus (0, -\infty)$ such that $z^\alpha = e^{\alpha \log|z|}$ on $z > 0$ and $F(\alpha, \beta, \gamma; t)$ is defined by

$$\sum_{k=0}^{\infty} \frac{(\alpha)_k(\beta)_k}{(1)_k(\gamma)_k} t^k, \quad (\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1).$$

The functions $p_i(t)$ satisfy the Gauss hypergeometric differential equation

$$\theta_t^2 - t(\theta_t + 1/2)^2, \quad \theta_t = t\frac{d}{dt}.$$

Moreover, $(p_1(t), p_2(t))$ is the fundamental set of solutions.

Next, we consider the following configuration and take the following cycles;

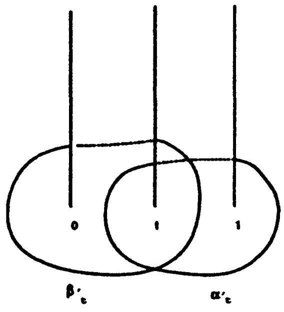

Figure 2.

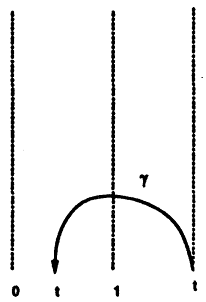

Figure 3.

We define the following period map;

$$(p_1'(t), p_2'(t)) = \left(\int_{\alpha_t'} \frac{dx}{y}, \int_{\beta_t'} \frac{dx}{y} \right).$$

These functions can also be expressed by the Gauss hypergeometric functions.

PROPOSITION 1.2 *Assume the configuration of Figure 2 and* $\operatorname{Im} t > 0$. *We have*

$$p_1'(t) = -2i\Gamma(1/2)^2 t^{-1/2} F(1/2, 1/2, 1; (t-1)/t)$$
$$p_2'(t) = -2\Gamma(\frac{1}{2})^2 F(1/2, 1/2, 1; t).$$

When the parameter t changes as in Figure 3, the cycles α_t and β_t are continuously deformed into α_t' and $\alpha_t' + \beta_t'$ respectively.

Therefore, we have the following connection formula.

PROPOSITION 1.3

$$\gamma^* \begin{pmatrix} p_1(t) \\ p_2(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} p_1'(t) \\ p_2'(t) \end{pmatrix}$$

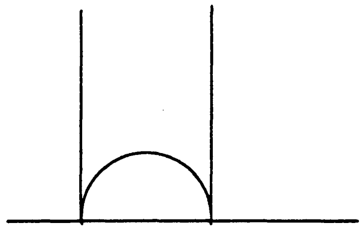

Figure 4

where γ^* *denotes the analytic continuation along the path* γ.

Here, let me mention the general definition of *connection formula*. Let Φ and Φ' be two fundamental sets of solutions of an ordinary differential equation of n-th order. Since the functions Φ and Φ' are the fundamental sets of solutions, there exists $n \times n$ matrix C such that

$$\Phi = C\Phi'.$$

The identity above is called the connection formula.

Using the connection formula, we can easily get the monodromy group of the period map.

PROPOSITION 1.4 *The monodromy of the period map is isomorphic to the discrete group* $\Gamma(2)$ *that is generated by*

$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^2, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^2, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We have studied the global behavior of the period map. It is also important to study the local behavior of the solutions of the Guass hypergeometric functions to determin the image.

PROPOSITION 1.5. (see, e.g., [IKSY; chapl]) *The set of the functions*

$$\phi_1 = F(1/2, 1/2, 1; t)$$

$$\phi_2 = \log t \cdot F(1/2, 1/2, 1; t) + O(t)$$

is the fundamental set of solutions of the Gauss hypergeometric equation $\theta_t^2 - t(\theta_t + 1/2)^2$ *around the point* $t = 0$.

Let $(q_1(t), q_2(t))$ be a fundamental set of real valued solutions of $\theta_t^2 - t(\theta_t + 1/2)^2$ on $(-\infty, 0)$. Then the image of $(-\infty, 0)$ by $q_2(t)/q_1(t)$ is a part of a line in $\mathbf{C}$. Since $p_2(t)/p_1(t)$ can be expressed by a linear fractional transformation of $q_2(t)/q_1(t)$, the image of the segment $(-\infty, 0)$ by $p_2(t)/p_1(t)$ is a part of a circle or a line in $\mathbf{C}$. Similarly, the image of the segments $(0,1)$ and $(1,\infty)$ by the map $p_2(t)/p_1(t)$ are parts of lines or circles in $\mathbf{C}$. So, the image of the upper half plane is a hyperbolic triangle enclosed by lines or circles. It follows from Proposition 1.5 and a similar argument at $t = 1$ that the each angle of the triangle is 0 (see Figure 4).

By virtue of the propositions and the observation above, we have the main theorem of the Gauss-schwartz theory.

THEOREM (Gauss, Schwartz) *Put $\tau = p_2(t)/p_1(t)$. The function τ is the multivalued function on $\mathbf{P}^1 \setminus \{0, 1, \infty\}$ and the image of the map is the upperhalf plane $H = \{\, z \mid \operatorname{Im} z > 0\}$. Moreover, the inverse map $\lambda(\tau)$ is the single-valued holomorphic function on H which satisfies*

$$\lambda(\tau + 2) = \lambda(\tau), \quad \lambda\left(\frac{\tau}{2\tau + 1}\right) = \lambda(\tau).$$

Although, the theorem above was found in the 19th century, Gauss-Schwartz type theorem and Torelli type theorem have been interested in even in our decades; Gauss-Schwartz type theorem has been studied by Terada, Deligne, Mostow, Kyoji Saito, Matsuzawa, Oyama, V.V.Vatyrev, Varchenko, Yoshida, Matsumoto, Sasaki and so on. Moreover, one of the motivation of the theory of the mixed Hodge structure is the Gauss-schwartz theory and Torelli's theorem. Unfortunately, it is beyond my ability to give an introduction to the theory. Anyway, one of the motivation of the study of local and global behaviors of the hypergeometric functions is the Gauss-schwartz theory. In the following sections, we focus on the study of local and global behavior of the $\mathcal{A}$-hypergeometric functions defined by Gel'fand, Zelevinsky and Kapranov.

Acknowledgement: I've learned a lot about the Gauss-schwartz theory from Prof. Keiji Matsumoto, who has nice works on the Gauss-schwartz theory for the hypergeometric functions on the Grassmann manifold ([MSY]). I would like to say many thanks to him.

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We have no space to give comprehensive references here. Please see the references of the papers above.

2. $\mathcal{A}$-hypergeometric system

Let us quickly review the theory of $\mathcal{A}$-hypergeometric system defined by Gel'fand, Zelevinsky and Kapranov ([GZK2]).

Let $\mathcal{A} = \{a_1, \ldots, a_n\}$ be a set of n-points in $\mathbf{Z}^d$ which satisfies the conditions:

(2.1) there exists a vector $c \in \mathbf{Z}^d$ such that

$$\langle c, a_i \rangle = 1, \quad i = 1, \ldots, n,$$

$$\mathbf{Z}a_1 + \cdots + \mathbf{Z}a_n = \mathbf{Z}^d. \tag{2.2}$$

We regard the a_i as the column vector and denote the (i,j)-element of the matrix $(a_1, \ldots, a_n)$ by a_{ij}. Let $\alpha_1, \ldots, \alpha_d$ be parameters. Put

$$p_i = \sum_{j=1}^{n} a_{ij} x_j \partial_j - \alpha_i, \quad i = 1, \ldots, d, \quad \partial_i = \frac{\partial}{\partial x_i}$$

and let

$$\mathcal{D}_{\mathcal{A}} = \mathcal{O}_{\mathcal{A}} \langle \partial_1, \ldots, \partial_n \rangle, \quad \mathcal{O}_{\mathcal{A}} = \mathcal{O}_n$$

be the sheaf of the differential operators on the $\mathcal{A}$-space $\mathbf{C}^n$. The $\mathcal{A}$-hypergeometric system $M_{\mathcal{A}}$ is defined by

$$M_{\mathcal{A}} = \mathcal{D}_{\mathcal{A}} / H_{\mathcal{A}}, \quad H_{\mathcal{A}} = \sum_{i=1}^{d} \mathcal{D}_{\mathcal{A}} p_i + I_{\mathcal{A}}$$

where $I_{\mathcal{A}}$ is the left ideal of $\mathcal{D}_{\mathcal{A}}$ generated by

$$\Delta_b = \prod_{b_j > 0} \partial_j^{b_j} - \prod_{b_j < 0} \partial_j^{-b_j}, \quad b = (b_1, \ldots, b_n) \in \ker(a_1, \ldots, a_n) \cap \mathbf{Z}^n.$$

EXAMPLE 2.1 Put

$$\mathcal{A} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

Then the left ideal $H_{\mathcal{A}}$ is generated by

$$\begin{aligned} p_1 &= x_3\partial_3 + x_4\partial_4 - \alpha_1 \\ p_2 &= x_1\partial_1 + x_3\partial_3 - \alpha_2 \\ p_3 &= x_2\partial_2 + x_4\partial_4 - \alpha_3 \\ &\partial_1\partial_4 - \partial_2\partial_3. \end{aligned} \tag{2.3}$$

The solution of this system can be expressed by using the Gauss hypergeometric function, because we can easily see that the function

$$\int_C (x_1 + x_3 z)^{\alpha_1}(x_2 + x_4 z)^{\alpha_2} z^{\alpha_3 - 1} dz$$

is the solution of the system of differential equations (2.3) ([GZK3; 260p]). We also note that the following function is the solutions of the system above;

$$x_1^{\alpha_2-\alpha_1} x_2^{\alpha_3} x_3^{\alpha_1} f(\alpha_2 - \alpha_1, \alpha_3, \alpha_1; z)$$

where

$$z = \frac{x_1 x_4}{x_2 x_3}$$

and

$$f(a,b,c;z) = \sum_{k=1}^{\infty} z^k / (\Gamma(a+k+1)\Gamma(b-k+1)\Gamma(c-k+1)\Gamma(k+1)).$$

We denote the sheaf of holomorphic solutions

$$\{f \in \mathcal{O} \mid \ell f = 0, \quad \forall \ell \in H_{\mathcal{A}}\}$$

by $\mathcal{H}om_{\mathcal{D}_{\mathcal{A}}}(M_{\mathcal{A}}, \mathcal{O}_{\mathcal{A}})$. In fact, any element h of $\mathcal{H}om_{\mathcal{D}_{\mathcal{A}}}(M_{\mathcal{A}}, \mathcal{O}_{\mathcal{A}})$ satisfies

$$\ell h(1) = h(\ell \cdot 1) = h(0) = 0, \quad \forall \ell \in H_{\mathcal{A}}.$$

So, $h(1)$ is the solution.

The left $\mathcal{D}_{\mathcal{A}}$-module $M_{\mathcal{A}}$ is holonomic system. The most fundamental result about the holonomic system is the following theorem due to Kashiwara.

THEOREM 2.1. (M.Kashiwara [K1]) *Let M be a holonomic system on $\mathbf{C}^n$. There exists a decomposition of $\mathbf{C}^n$ into analytic sets $\cup X_\mu$ such that the sheaf*

$$\mathcal{H}om_{\mathcal{D}_n}(M, \mathcal{O}_n)_{|X_\mu}$$

is locally constant sheaf of finite rank; the sheaf $\mathcal{H}om_{\mathcal{D}_n}(M, \mathcal{O}_n)$ is called the constructible sheaf of finite rank.

In order to convice you this theorem, let me give you an example.

EXAMPLE 2.2. Put $n = 1$ and consider the holonomic system

$$\mathcal{D}_1/\mathcal{D}_1 p, \quad p = \theta_x^2 - x(\theta_x + 1/2)^2, \quad \theta_x = x\frac{d}{dx}, \mathcal{D}_1 = \mathcal{O}_1\langle d/dx \rangle.$$

We decompose $\mathbf{C}^1$ into the sets

$$X_\alpha = \mathbf{C} \setminus \{0,1\}, X_\beta = \{0\}, X_\gamma = \{1\}.$$

It follows from Proposition 1.5 and a similar argument around the point $x = 1$ that the sheaf

$$\mathcal{H}om_{\mathcal{D}_1}(\mathcal{D}_1/\mathcal{D}_1 p, \mathcal{O}_1), \quad \mathcal{O}_1 = \mathbf{C}\{x\}$$

is locally constant sheaf of rank 2 on X_α, of rank 1 on X_β and of rank 1 on X_γ respectively.

There exists an open dense stratum in the stratification $\cup X_\mu$. The stratum is called the *generic stratum*. Gel'fand, Zelevinsky and Kapranov proved the following theorem.

THEOREM 2.2. ([GZK2,GZK1]) *Let $M_\mathcal{A}$ be the $\mathcal{A}$-hypergeometric system. The generic stratum $X'_\mathcal{A}$ is the complement of the zero set of the principal $\mathcal{A}$-determinant $E_\mathcal{A}$. Moreover, the solution sheaf is the locally constant sheaf of rank* vol$(\mathcal{A})$ *on the generic stratum $X'_\mathcal{A}$.*

When we look at these 2 theorems, a natural question arises; study the $\mathcal{A}$-hypergeometric system on the non-generic stratas. Gel'fand, Kapranov and Zelevinsky gave an answer to this question in a quite abstract way; they express the solution sheaf by the twisted cohomology ([GZK3; 270p, line 9]). Here, we will give a description of the structure of the solution sheaf by using the secondary polytope in an elementary way.

3. Secondary polytope

Let $(\omega_1, \ldots, \omega_n)$ be a vector in $\mathbf{R}^n$. Consider the convex hull H of the points

$$\{(a_1, \omega_1), \ldots, (a_n, \omega_n)\}$$

where a_i are vectors in $\mathbf{Z}^d$. Let

$$\pi : \mathbf{R}^{d+1} \ni (y_1, \ldots, y_{d+1}) \longmapsto (y_1, \ldots, y_d) \in \mathbf{R}^d$$

be the projection. The projection by π of the lower part of the convex hull H induces the polyhedral subdivision of conv$(\mathcal{A})$. The polyhedral subdivision obtained by this way is called the *regular* polyhedral subdivision. When the polyhedral subdivision is the triangulation of $\mathcal{A}$, the polyhedral subdivision is called the *regular* triangulation. The set of all regular polyhedral subdivisions is poset (partially ordered set) by the refinement.

Let T be the set of all triangulations of $\mathcal{A}$. Here, by the triangulation of $\mathcal{A}$, we mean a triangulation of conv$(\mathcal{A})$ of which vertices are in $\mathcal{A}$. The secondary polytope $\Sigma(\mathcal{A})$ is defined by

$$\Sigma(\mathcal{A}) = \mathrm{conv}_{\Delta \in T} \phi_\Delta, \quad \phi_\Delta = \sum_{\tau \in \Delta} \mathrm{vol}(\tau)(e_{\tau_1} + \cdots + e_{\tau_d}) \in \mathbf{R}^n$$

where e_i denotes the i-th standard basis vector in $\mathbf{R}^n$.

THEOREM 3.1. ([GZK1], see also [BFS]) *The face lattice of $\Sigma(\mathcal{A})$ is anti-isomorphic to the poset of the all regular polyhedral subdivisions of $\mathcal{A}$. Especially, the vertices of $\Sigma(\mathcal{A})$ are in one-to-one correspondence with the regular triangulations.*

An algorithm of enumerating all regular triangulations is given by [BFS].

EXAMPLE 3.1. The k-simplex Δ_k is the convex hull of

$$e_1, \ldots, e_{k+1}$$

where e_i denotes the i-th standard basis vector of $\mathbf{R}^{k+1}$. We consider the general prism $\Delta_1 \times \Delta_{n-1}$ in $\mathbf{R}^2 \oplus \mathbf{R}^n = \mathbf{R}^{2+n}$ of which vertices are

$$e_i \oplus e_j, \quad (i = 1, 2, 1 \leq j \leq n, e_i \in \mathbf{R}^2, e_j \in \mathbf{R}^n).$$

Let

$$\tau^{(i)} = \{(1,1), (1,2), \ldots, (1, n-i+1), (2, n-i+1), (2, n-i+2), \ldots, (2, n)\}$$

be the n-simplex where (p, q) denotes the point $e_p \oplus e_q$. The collection

$$T = \{\tau^{(1)}, \ldots, \tau^{(n)}\}$$

is a triangulation of $\mathcal{A}_n$ and will be cited as the stair-case triangulation. The n-simplex $\tau^{(i)}$ is often figured, for example in case of $n = 4$, as follows;

$$\tau^{(1)} = \begin{pmatrix} 11 & 12 & 13 & 14 \\ & & & 24 \end{pmatrix}, \quad \tau^{(2)} = \begin{pmatrix} 11 & 12 & 13 & \\ & & 23 & 24 \end{pmatrix}, \tau^{(3)} = \begin{pmatrix} 11 & 12 & & \\ & 22 & 23 & 24 \end{pmatrix}, \tau^{(4)} = \begin{pmatrix} 11 & & & \\ 21 & 22 & 23 & 24 \end{pmatrix}.$$

Let us note that the general prism $\Delta_1 \times \Delta_{n-1}$ admits the action of the group of all permutations of n-letters $\mathfrak{S}_n$;

$$\sigma : \Delta_1 \times \Delta_{n-1} \ni e_i \oplus e_j \longmapsto e_i \oplus e_{\sigma(j)} \in \Delta_1 \times \Delta_{n-1}, \quad \sigma \in \mathfrak{S}_n.$$

So, we get $n!$ triagulations $\{T^\sigma\}$. All triangulations of $\Delta_1 \times \Delta_{n-1}$ can be obtained in this way. Moreover, they are regular triangulations. Specializing the result of [BFS], we have the following result.

PROPOSITION 3.1 ([BFS]) *The secondary polytope $\Sigma(\Delta_1 \times \Delta_{n-1})$ is $(n-1)$-dimensional zonotope which is the Minkowskii sum of $\binom{n}{2}$ segments.*

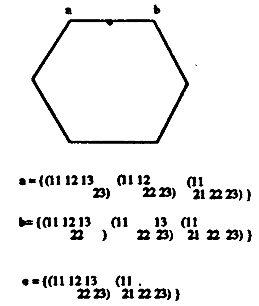

Figure 5.

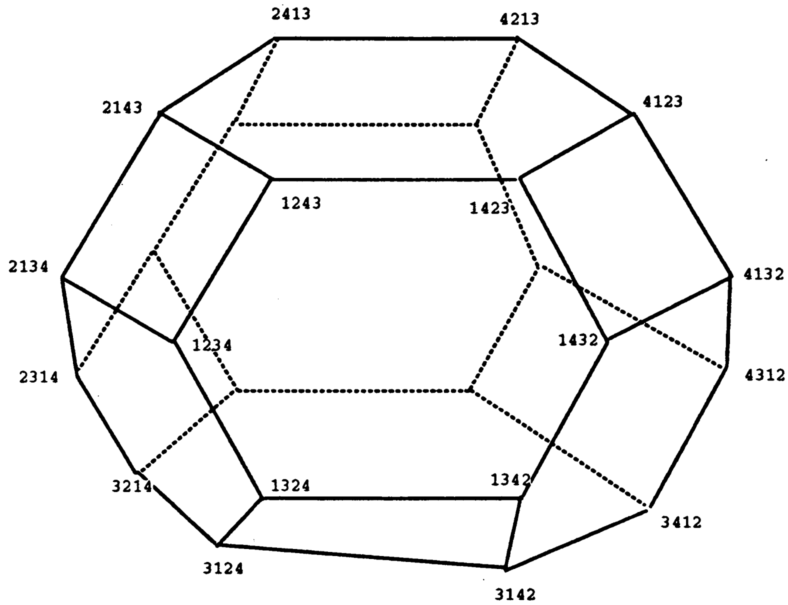

Figure 6.

We illustrate Theorem 3.1 and Proposition 3.1 in case of $n = 3$ and $n = 4$.

4. Formal restriction

Let $\{\Gamma^{(1)}, \Gamma^{(2)}, \ldots\}$ be a regular polyhedral subdivision of $\mathcal{A}$ and we fix it. We assume $\Gamma^{(1)} = \{1, \ldots, m\}$ by changing the indices of vertices and put $\Gamma = \Gamma^{(1)}$. Let M_Γ be the hypergeometric $\mathcal{D}$-module defined by Γ on $\mathbf{C}^m$. We will describe the solution sheaf of $M_{\mathcal{A}}$ on a non-generic stratum by using the M_Γ, which we will call the *formal restriction*.

Put

$$X_\Gamma = \{x \mid x_{m+1} = \cdots = x_n = 0\}$$

and let

$$j\ :\ X_\Gamma \longrightarrow X_{\mathcal{A}} = \mathbf{C}^n$$

be the embedding. The restriction of $M_{\mathcal{A}}$ to X_Γ as $\mathcal{D}$-moudle ([K1]) is defined by

$$j^* M_{\mathcal{A}} = j^{-1}(\mathcal{D}_{\mathcal{A}}/(H_{\mathcal{A}} + \sum_{i=1}^{m} x_i \mathcal{D}_{\mathcal{A}}).$$

Note that there exists a natural morphism from M_Γ to $j^* M_{\mathcal{A}}$, because $H_\Gamma \subseteq H_{\mathcal{A}} + \sum_{i=1}^m x_i \mathcal{D}_{\mathcal{A}}$. The natural morphism is the isomorphism on the generic stratum on X_Γ.

THEOREM 4.1. ([T1]) *Let F_τ be the minimal integral supporting function of the facet τ of the cone spanned by Γ.*

(a) *Suppose the conditions*

(1) $\sum_{i=1}^m \mathbf{Z} a_i = \mathbf{Z}^d$,

(2) (normality) $\sum_{i=1}^m \mathbf{Z}_{\geq 0} a_i = (\sum_{i=1}^m \mathbf{R}_{\geq 0} a_i) \cap \mathbf{Z}^d$,

(3) $F_\tau(\alpha) \notin \mathbf{Z}_{\geq 0}$ *for all facets τ of the cone spanned by Γ,*

are satisfied, then the morphism

$$r\ :\ M_\Gamma \longrightarrow j^* M_{\mathcal{A}}$$

is surjective.

(b) *Let T be a regular triangulation which is a refinement of the regular polyhedral subdivision $\cup\Gamma^{(k)}$. If the parameter α is T-nonresonant and the conditions* (1), (2), (3) *are satisfied, then we have the isomorphism*

$$\mathcal{H}om_{\mathcal{D}_{\mathcal{A}}}(M_{\mathcal{A}}, \mathcal{O}_{\mathcal{A}})_{|X_\Gamma} = \mathcal{H}om_{\mathcal{D}_\Gamma}(j^* M_{\mathcal{A}}, \mathcal{O}_\Gamma) = \mathcal{H}om_{\mathcal{D}_\Gamma}(M_\Gamma, \mathcal{O}_\Gamma)$$

on the generic stratum of X_Γ. Moreover, we have

$$M_\Gamma = j^* M_{\mathcal{A}}$$

on the generic stratum of X_Γ.

In [T1], the condition (3) is given by using the b-function defined by Mutsumi Saito ([S1]). M.Saito kindly told me that the condition can be expressed by using the supporting function of the cone. Moreover, he sent me the proof of Theorem 4.1 without the normality condition during the preparation of this exposition (September 2, 1993).

EXAMPLE 4.1. Put $\Gamma = \Delta_1 \times \Delta_{n-1} \setminus \{(1,n)\}$. The decomposition

$$\Gamma \cup \begin{pmatrix} & & & 1n \\ 21 & 22 & \cdots & 2n \end{pmatrix}$$

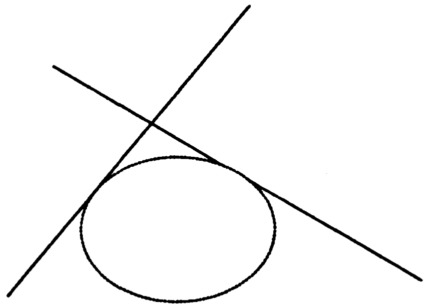

Figure 7.

is the regular polyhedral subdivision. Γ is the cone over $\Delta_1 \times \Delta_{n-2}$ and

$$u_{2n}^{\alpha_n - 1} M_\Gamma u_{2n}^{-\alpha_n + 1} = M_{\Delta_1 \times \Delta_{n-2}}.$$

So, the solution sheaf of the $\Delta_1 \times \Delta_{n-1}$-hypergeometric system is isomorphic to the solution sheaf of the $\Delta_1 \times \Delta_{n-2}$-hypergeometric system on the hyperplane $u_{1n} = 0$. Here, we denote the independent variables by u_{ij} and the parameters by $(-\beta_2, \alpha_1 - 1, \ldots, \alpha_n - 1)$. Note that the generic stratum of $M_{\Delta_1 \times \Delta_{n-1}}$ is given in Proposition 5.1.

EXAMPLE 4.2. We, again, consider the $\mathcal{A}$-hypergeometric system of the general prism $\Delta_1 \times \Delta_{n-1}$. The line defined by the origin and the point $(1, n)$ is a face of the cone defined by the general prism. We denote the torus orbit corresponding to the line by O_v. The normal bundle $T^*_{O_v}\mathbf{C}^{2n}$ is an irreducible component of the characteristic variety of the $\Delta_1 \times \Delta_{n-1}$-hypergeometric system and the multiplicity is 1 by virtue of [GZK2]. So, the index of the hypergeometric system is $n-1$ at a generic point x_0 in $u_{1n} = 0$. It follows from the index theorem of Kashiwara that we have

$$n - 1 = \sum_{i=0}^{2n} \dim_{\mathbf{C}} (-1)^i \mathcal{E}xt^i_{\mathcal{D}_\mathcal{A}} (M_{\Delta_1 \times \Delta_{n-1}}, \mathcal{O})_{x_0}.$$

On the other hand,

$$\dim_{\mathbf{C}} \mathcal{H}om_\mathcal{A}(M_{\Delta_1 \times \Delta_{n-1}}, \mathcal{O})_{x_0} = n - 1$$

from Example 4.1 and

$$\dim_{\mathbf{C}} \mathcal{E}xt^i_{\mathcal{D}_\mathcal{A}}(M_{\Delta_1 \times \Delta_{n-1}}, \mathcal{O})_{x_0} = 0$$

for $i \geq 2$ because of the regular holonomicity of the system ([Hot]). Therefore, the first cohomology $\mathcal{E}xt^1_{\mathcal{D}_\mathcal{A}}(M_{\Delta_1 \times \Delta_{n-1}}, \mathcal{O})$ also vanishes on the generic stratum of $u_{1n} = 0$.

We have studied a structure of the constructible sheaf $\mathcal{H}om_{\mathcal{D}_\mathcal{A}}(M_\mathcal{A}, \mathcal{O}_\mathcal{A})$. Our study can be applied to derivation of connection formulas among hypergeometric series. Gel'fand, Zelevinsky and Kapranov showed that each regular triangulation of $\mathcal{A}$ determines a fundamental set of solutions expressed by series; we can attach a set of series solutions to each vertex of the secondary polytope. So, it is a natural question to find connection formulas among them. It is very difficult to find them in general case, because the fundamental groupoid of the generic stratum $X'_{\mathcal{A}}$ is unknown. Fortunately, the topology of the generic stratum of the $\Delta_1 \times \Delta_{n-1}$-hypergeometric system is relatively easy. We can explicitly derive connection formulas in that case.

Let $\mathcal{F}$ be a field and suppose that a group G acts on $\mathcal{F}$. A set of matrices $\{C(g) \in GL(m, \mathcal{F}) \,|\, g \in G\}$ that satisfies the condition

$$C(gh) = C(h)C(g)^h, \quad g, h \in G$$

is called *the multiplicative* 1*-cocycle* of the group G.

The connection formulas of the $\Delta_1 \times \Delta_{n-1}$-hypergeometric functions can be expressed by those of the $\Delta_1 \times \Delta_{n-2}$-hypergeometric functions and the set of the formulas is given as a multiplicative 1-cocycle of $\mathfrak{S}_n$, where we can understand $\mathfrak{S}_n$ as the group generated by restructurings of triangulations.

5. Connection formulas of the $\Delta_1 \times \Delta_{n-1}$-hypergeometric function

Put

$$\chi_{1j} = 0 \oplus e_j, \quad \chi_{2j} = 1 \oplus e_j, \quad j = 1, \dots, n$$

and

$$\mathcal{A}_n = \{\chi_{11}, \dots, \chi_{1n}, \chi_{21}, \dots, \chi_{2n}\} = \Delta_1 \times \Delta_{n-1}.$$

We consider the $\Delta_1 \times \Delta_{n-1}$-hypergeometric system $M_{\mathcal{A}_n}$ with the parameter $(-\beta_2, \alpha_1 - 1, \dots, \alpha_n - 1)$. We denotes the independent variables by

$$u = \begin{pmatrix} u_{11} & \cdots & u_{1n} \\ u_{21} & \cdots & u_{2n} \end{pmatrix}$$

to clarify the symmetry of the system. We can easily see the following from Theorem 2.2.

PROPOSITION 5.1. *Put*

$$X'_{\mathcal{A}_n} = \left\{ u \in \mathbf{C}^{2n} \,\middle|\, \prod_{i=1,2, j=1,\dots,n} u_{ij} \prod_{1 \le k < \ell \le n} \begin{vmatrix} u_{1k} & u_{1\ell} \\ u_{2k} & u_{2\ell} \end{vmatrix} \neq 0 \right\}.$$

Then

$$\dim_{\mathbf{C}} \mathcal{H}om_{\mathcal{D}_{\mathcal{A}_n}}(M_{\mathcal{A}_n}, \mathcal{O}_{\mathcal{A}_n}) = \mathrm{vol}(\Delta_1 \times \Delta_{n-1}) = n \quad \text{on} \quad X'_{\mathcal{A}_n}.$$

Next, following the method of [GZK2], we give the fundamental set of solutions expressed by series determined by the stair case triangulation T. Let $\gamma^{(\tau)} \in \mathbf{C}^{2n}$ be the solution of the linear equation

$$A\gamma^{(\tau)} = \begin{pmatrix} -\beta_2 \\ \alpha_1 - 1 \\ \cdot \\ \cdot \\ \cdot \\ \alpha_n - 1 \end{pmatrix}, \quad A = (\chi_{11}, \dots, \chi_{1n}, \chi_{21}, \dots, \chi_{2n})$$

with the constraint

$$\gamma_i^{(\tau)} = 0 \quad \text{when } i \notin \tau \in T.$$

Define series

$$\phi_\tau = \sum_{k \in \ker A \cap \mathbf{Z}^{2n}} u^{\gamma^{(\tau)}+k} / \prod_{i=1}^{2n} \Gamma(\gamma_i^{(\tau)} + k_i + 1).$$

Specializing the result of [GZK2; Th3,5], we have the following.

PROPOSITION 5.2. *If the stair-case triangulation is T-nonresonant, then*

$$\{\phi_\tau \mid \tau \in T\}$$

is a fundamental set of solutions of $M_{\mathcal{A}_n}$.

The functions ϕ_τ are defined on a small open set. We will define an analytic continuation of the function to larger domains. In order to do it, we decompose $\mathbf{C}^{2n}$ into simply connected domains.

Let us denote the coordinates of $\mathbf{R}^{2n}$ by $\{\theta_{ij}\}$. We consider the hyperplane arrangement in $\mathbf{R}^{2n}$ defined by

$$(5.1) \qquad \begin{cases} \theta_{ij} = -\pi, 0, \pi, \\ \theta_{1i} - \theta_{2i} = \pm k\pi, \quad (k = 0, 1, 2) \\ (\theta_{1i} - \theta_{2i}) - (\theta_{1j} - \theta_{2j}) = \pm k\pi, \quad (k = 0, 1, 2, 3, 4). \end{cases}$$

We denote the set of maximal dimensinal cells that are contained in the domain

$$\{(\theta)_{ij} \mid -\pi < \theta_{ij} < \pi\}$$

by $\mathcal{S}$. For $s \in \mathcal{S}$, put

$$D(s) = \{(r_{ij} e^{i\theta_{ij}}) \mid \theta_{ij} \in s, r_{ij} > 0\}.$$

The domain $D(s)$ is simply connected and is contained in the generic stratum $X'_{\mathcal{A}_n}$. We can define unique analytic continuation of the function ϕ_τ to $D(s)$, which we denote by φ_τ. Put

$$\Phi = (\varphi_{\tau(1)}, \dots, \varphi_{\tau(n)}), \quad \Phi^\sigma = (\varphi_{\tau(1)^\sigma}, \dots, \varphi_{\tau(n)^\sigma}), \quad \sigma \in \mathfrak{S}_n.$$

The function Φ^σ is also the fundamental set of solutions of the system $M_{\mathcal{A}_n}$. Define the connection matrix $C(\sigma)$ by

$$\Phi = C(\sigma)\Phi^\sigma.$$

It follows from the definition that the matrix $C(\sigma)$ is constant on each $D(s)$. So, the matrix $C(\sigma)$ is the Heaviside function on the hyperplane arrangement (5.1). The set of matrices $\{C(\sigma)\}$ is the multiplicative 1-cocycle of $\mathfrak{S}_n$ and they can be expressed as follows;

Theorem 5.1. ([T1]) *Assume the T-nonresonant condition and the condition $\alpha_i, \beta_j \notin \mathbf{Z}$. Define $p \times p$ matrix C_p by the recurrence relations*

$$C_p(s_i;\alpha_1,\dots,\alpha_p;\beta_1,\beta_2;1,\dots,p) = 1 \oplus C_{p-1}(s_i;\alpha_1,\dots,\alpha_{p-1};\beta_1,\beta_2+\alpha_p-1;1,\dots,p-1)$$

for $1 \le i < p-1$,

$$C_p(s_{p-1};\alpha_1,\dots,\alpha_p;\beta_1,\beta_2;1,\dots,p) = C_{p-1}(s_{p-2};\alpha_2,\dots,\alpha_p;\beta_1+\alpha_1-1,\beta_2;2,\dots,p)\oplus 1$$

and

$$C_2(s_1;\alpha_1,\alpha_2;\beta_1,\beta_2;i,j) = \begin{pmatrix} q_{ij}(\alpha_1,\beta_2) & q_{ij}(-\beta_2,-\alpha_1) \\ q_{ij}(-\beta_1,-\alpha_2) & q_{ij}(\alpha_2,\beta_1) \end{pmatrix}$$

where $\sum_{i=1}^n \alpha_i + \beta_1 + \beta_2 = n$ and

$$q_{ij}(\alpha_1,\beta_2) = \frac{\frac{[ij]^{-2\alpha_1}}{(-[ij])^{-2\alpha_1}} - 1}{\frac{[ij]^{-2\alpha_1-2\beta_2}}{(-[ij])^{-2\alpha_1-2\beta_2}} - 1}[ij]^{-\beta_2}\frac{u_{1j}^{\beta_2}u_{2i}^{\beta_2}}{u_{1i}^{\beta_2}u_{2j}^{\beta_2}}, \quad [ij] = \frac{u_{1j}u_{2i}}{u_{1i}u_{2j}}.$$

Then, the matrix

$$C_n(s_i;\alpha_1,\dots,\alpha_n;\beta_1,\beta_2;1,\dots,n)$$

is the connection matrix among the solutions Φ and Φ^{s_i} where $s_i = (i,i+1) \in \mathfrak{S}_n$.

Notice that the function $q_{ij}(a,b)$ is the Heaviside function defined on the hyperplane arrangement (5.1).

The proof of this theorem is based on Theorem 4.1. In order to explain how to use the description on the constructible sheaf (e.g. Theorem 4.1) to prove functional identities of hypergeometric functions, we, finally, show you a small example.

Example 5.1.

Problem: Prove the identity

$$f(\alpha_2-\alpha_1,\alpha_3,\alpha_1;z) = \frac{\Gamma(\alpha_1+1)}{\Gamma(-\alpha_2+1)}(1-z)^{\alpha_3} f\left(\alpha_2-\alpha_1,\alpha_3,-\alpha_2;\frac{z}{z-1}\right)$$

where

$$z = \frac{x_1 x_4}{x_2 x_3}$$

and

$$f(a,b,c;z) = \sum_{k=1}^{\infty} z^k / (\Gamma(a+k+1)\Gamma(b-k+1)\Gamma(c-k+1)\Gamma(k+1)).$$

Answer: We can see that the each side $\times x_1^{\alpha_2-\alpha_1} x_2^{\alpha_3} x_3^{\alpha_1}$ of the formula above satisfies the $\mathcal{A}$-hypergeometric system of Example 2.1 by a little tedious or by a clever way (pull up the functions on the Grassmann manifold $G(2,4)$). It follows from Theorem 4.1 that

$$\dim_{\mathbf{C}} \mathcal{H}om_{\mathcal{D}_{\mathcal{A}_2}}(M_{\mathcal{A}_2}, \mathcal{O}_{\mathcal{A}_2})|_{x_4=0} \leq 1.$$

So, it is enough to prove the formula on $x_4 = 0$ and it can be easily checked. $\square$

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